Instructional Strategies

The purpose of this chapter is not to prescribe the usage of any particular instructional strategies, but to enhance teaching repertoire. Teachers have a wide choice of instructional strategies for any given lesson, and effective teachers look for a fit between the material to be taught and strategies to teach it. (See grade level chapters for more specific examples.) Ultimately, teachers and administrators must decide which instructional strategies are most effective in addressing the unique needs of individual students.

In a standards-based curriculum, effective lessons, units, or modules are carefully developed and are designed to engage all members of the class in learning activities focused on student mastery of specific standards. Such lessons, lasting usually at least 50 to 60 minutes daily (excluding homework), connect the standards to the basic question of why mathematics is relevant, true, and important. Central to the CCSSM and this framework is the goal that all students should be college and career ready by mastering the standards. Lessons need to be designed so that students are constantly being exposed to new information while building conceptual understanding, practicing skills, and reinforcing their mastery of information introduced previously. The teaching of mathematics must be carefully sequenced and organized to ensure that all standards are taught at some point and that prerequisite skills form the foundation for more advanced learning, yet it should not proceed in a strict linear order, requiring students to master each standard completely before being introduced to the next. Practice leading toward mastery can be embedded in new and challenging problems promoting conceptual understanding and fluency in mathematics.

Key Instructional Shifts

The three major principles on which the CCSSM are based are focus, coherence and rigor. As teachers work to incorporate these shifts into their practice, focus on these areas can help schools and districts develop a common understanding of what is necessary for mathematics instruction as they move forward with the implementation of CCSSM.

Focus. Focus requires that we significantly narrow the scope of content in each grade so that students more deeply experience that which remains. Administrators and teachers are cautioned that instructional time is finite and that focus compromised is focus destroyed.

The overwhelming focus of the CCSSM in early grades is arithmetic, along with the components of measurement that support it. That includes the concepts underlying arithmetic, the skills of arithmetic computation, and the ability to apply arithmetic to solve problems and put arithmetic to engaging uses. Arithmetic in the K–5 standards is an important life skill, as well as a thinking subject and a rehearsal for working with algebraic concepts in the middle grades.

Focus remains important through the middle and high school grades in order to prepare students for college and careers; surveys suggest that postsecondary instructors value greater mastery of prerequisites over shallow exposure to a wide array of topics with dubious relevance to postsecondary work.

***Coherence*.** Coherence is about making math make sense. Mathematics is not a list of disconnected tricks or mnemonics. It is an elegant subject in which powerful knowledge results from reasoning with a small number of principles such as place value and properties of operations. The standards define progressions of learning that leverage these principles as they build knowledge over the grades.

When people talk about coherence, they often talk about making connections between topics. The most important connections are vertical: the links from one grade to the next that allow students to progress in their mathematical education. That is why it is critical to think across grades and examine the progressions in the standards to see how major content develops over time.

Connections at a single grade level can be used to improve focus, by tightly linking secondary topics to the major work of the grade. For example, in grade 3, bar graphs are not “just another topic to cover.” Rather, the standard about bar graphs asks students to use information presented in bar graphs to solve word problems using the four operations of arithmetic. Instead of allowing bar graphs to detract from the focus on arithmetic, the standards are showing how bar graphs can be positioned in support of the major work of the grade. In this way coherence can support focus.

Rigor**.** To help students meet the expectations of the CCSSM, educators need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skill and fluency, and applications. The word “understand” is used in the Standards to set explicit expectations for conceptual understanding, the word “fluently” is used to set explicit expectations for fluency, and the phrase “real-world problems” and the star symbol (★) are used to set expectations and flag opportunities for applications and modeling (which is a standard for mathematical practice as well as a conceptual category in higher mathematics). Real-world problems and standards that support modeling are also opportunities to provide activities related to careers and the work-world.

To date, curricula have not always been balanced in their approach to these three aspects of rigor. Some curricula stress fluency in computation, without acknowledging the role of conceptual understanding in attaining fluency. Some stress conceptual understanding, without acknowledging that fluency requires separate classroom work of a different nature. Some stress pure mathematics, without acknowledging that applications can be highly motivating for students, and moreover, that a mathematical education should prepare students for more than just their next mathematics course. At another extreme, some curricula focus on applications, without acknowledging that math doesn't teach itself. The CCSSM do not take sides in these ways, but rather they set high expectations for all three components of rigor in the major work of each grade.

Conceptual Understanding. Teachers need to teach more than "how to get the answer" and instead should support students' ability to access concepts from a number of perspectives so that students are able to see mathematics as more than a set of mnemonics or discrete procedures. Students demonstrate solid conceptual understanding of core mathematical concepts by applying them to new situations as well as writing and speaking about their understanding.

The focus provided by the CCSSM allows teachers and students to have the time and space to develop solid conceptual understanding. In return, the CCSSM require a real commitment to understanding. For example, it is not sufficient for students to simply know the procedure for finding equivalent fractions; they also need to know what it means for numbers to be written in equivalent forms. Attention to conceptual understanding allows students to build on prior knowledge.

Procedural Skills and Fluency. Teachers structure class time and/or homework time for students to practice procedural skills. Students develop fluency in core functions, such as addition, subtraction, multiplication, and division, so that they are able to understand and manipulate more complex concepts.

Note that fluency is not memorization absent understanding. It is the outcome of a carefully laid out learning progression that requires planning and practice. Procedural and computational fluencies imply accuracy with reasonable speed and refer to knowledge of procedures, when and how to use procedures appropriately, and skill and confidence in performing them accurately and efficiently.

Application. The CCSSM require application of mathematical concepts and procedures throughout all grades. Students are expected to use mathematics and choose the appropriate concepts for application even when they are not prompted to do so. Teachers should provide opportunities in all grade levels for students to apply mathematical concepts in real-world situations as it motivates students to learn mathematics and enables them to transfer this knowledge into their daily lives and future careers. Teachers in content areas outside of mathematics, particularly science, ensure that students are using grade-level appropriate mathematics to make meaning of and access content.

Students need to be given opportunities to gain deep insight into the mathematical concepts they are using and also develop fluency with the procedures that will be applied in these situations. Application without conceptual knowledge and procedural fluency makes problem solving substantially more difficult. Application can be motivational and interesting, and there is a need for students at all levels to connect the mathematics they are learning to the world around them (Adapted from Achieve the Core 2012 and PARCC 2012).

Standards for Mathematical Practice

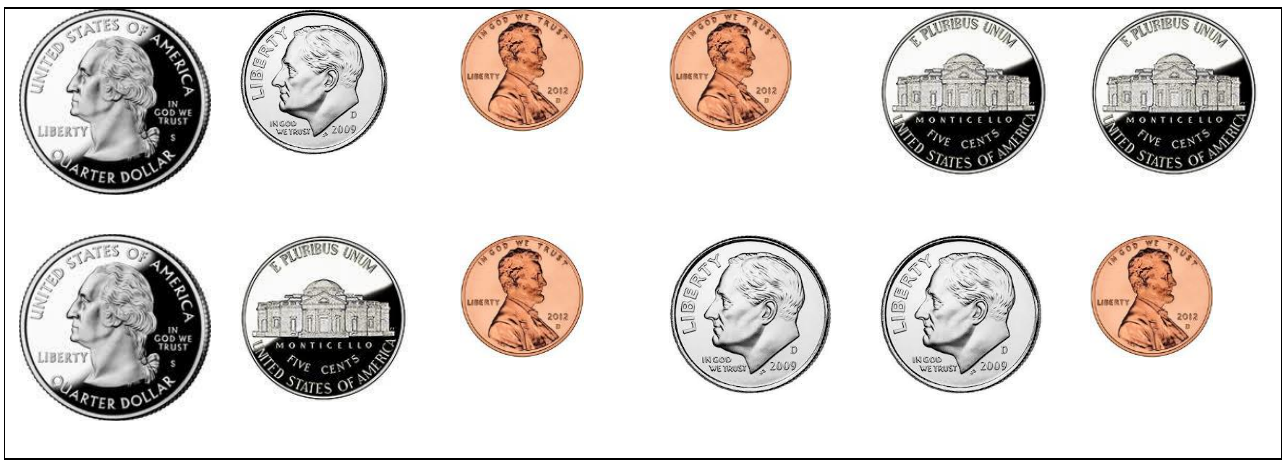
The Standards for Mathematical Practice (MP) describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy) (CCSSI 2012). Teachers need to design their instruction in order to effectively incorporate these standards. For example, teachers need to closely analyze their curriculum and identify the areas where content and practice standards intersect.

[Note: Graphic with the MP standards will be inserted here.]

The following curricular examples illustrate the types of problems incorporating the MP standards.

The problem below entitled *Migdalia's Savings,* addresses grade two standards 2.OA.1., 2.MD.8 and Mathematical Practice Standards MP 1, MP 4, MP 5, and MP 6. The problem requires students to count a combination of coins and then demonstrate understanding of subtracting money amounts by writing a story problem that shows how Migdalia spends her money.

Migdalia's Savings. Migdalia has worked really hard to save this much money, and now she gets to go to the store. How much money does Migdalia have? Write a story problem that shows how Migdalia spends her money. Did she have any money left?



This problem demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (MP 1) by choosing the strategies to use. They apply the mathematics they know to solve problems arising in everyday life (MP 4); utilize available tools such as concrete models; and use mathematically precise vocabulary to communicate their explanations through writing a story problem (MP 6).

Understanding Perimeter. The following hands-on activity illustrates the third grade standard 3.MD.8 as well as multiple Standards for Mathematical Practice (MP): Students will solve problems with fixed area and perimeter and develop an understanding of the concept of perimeter by walking around the perimeter of a room, using rubber bands to represent the perimeter of a plane figure on a geoboard, or tracing around a shape on an interactive whiteboard. They find the perimeter of objects; use addition to find perimeters; and recognize the patterns that exist when finding the sum of the lengths and widths of rectangles. Students use geoboards, tiles, and graph paper to find all the possible rectangles that have a given area (e.g., find the rectangles that have an area of 12 square units.) Once students have learned to find the perimeter of a rectangle, they record all the possibilities using dot or graph paper (MP 1), compile the possibilities into an organized list or a table (see below) (MP 4), and determine whether they have all the possible rectangles (MP 2). The patterns in the chart allow the students to identify the factors of 12, connect the results to the commutative property (MP 7), and discuss the differences in perimeter within the same area (MP 3). This chart can also be used to investigate rectangles with the same perimeter. It is important to include squares in the investigation.

Area (square inches)	Length (inches)	Width (inches)	Perimeter (inches)
12	1	12	26
12	2	6	16
12	3	4	14
12	4	3	14

12	6	2	16
12	12	1	26

(KATM 2012, 3rd FlipBook)

After School Job. This problem addresses content standards 4.OA.5 and 5.OA.3 and Mathematics Practice Standards MP 1, MP 3, MP 4, MP 5, and MP 6:

Leonard needed to earn some money so he offered to do some extra chores for his mother after school for two weeks. His mother was trying to decide how much to pay him when Leonard suggested the idea:

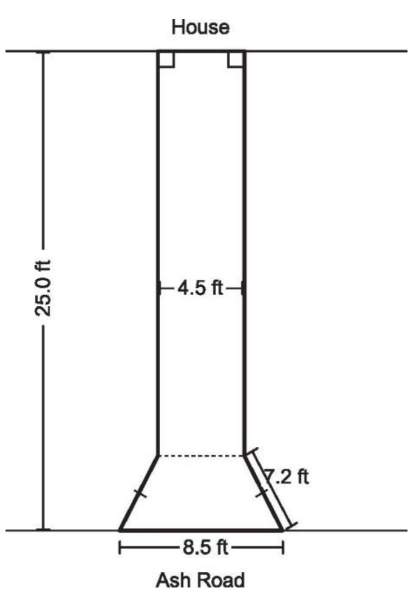
"Either you pay me $1.00 every day for the two weeks, or you can pay me 1¢ for the first day, 2¢ for the second day, 4¢ for the third day, and so on, doubling my pay every day."

Which option does Leonard want his mother to choose? Write a letter to Leonard's mother suggesting the option that she should take. Be sure to include pictures that explain that will explain your mathematical thinking.

The problem requires students to generate two numerical patterns using two given rules: "add 1" and "double the sum," generate terms in the resulting sequences over a 14 day time period, and explain why the first option would cost Leonard's mother much less money. This problem demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (MP1) by choosing the strategies to use. They make conjectures and build a logical progression through careful analyses (MP 3); apply the mathematics they know to solve problems arising in everyday life that are motivating to them (MP4); utilize available tools such as concrete models and calculators (MP 5); and use mathematically precise vocabulary to communicate their explanations through writing and through graphics such as charts (MP 6).

The following problem entitled *Ms. Olsen's Sidewalk* (*SBAC Appendix C*, Dec. 7, 2011), addresses content standards 7.G.6, 7.NS.3, 8.G.7 and Mathematical Practice Standards MP1, MP4, and MP6. In this task the students are given a real-world problem whose solution involves determining the areas of two-dimensional shapes as part of calculating the cost of a sidewalk.

Ms. Olsen's Sidewalk. Ms. Olsen is having a new house built on Ash Road. She is designing a sidewalk from Ash Road to her front door. Ms. Olsen wants the sidewalk to have an end in the shape of an isosceles trapezoid, as shown in the diagram.



The contractor charges a fee of $200 plus $12 per square foot of sidewalk. Based on the diagram, what will the contractor charge Ms. Olsen for her sidewalk? Show your work or explain how you found your answer.

A common problem with the calculation of the areas of trapezoids is the misuse of the length marked 7.2 ft. Students need to make use of this dimension, but must avoid falling into multiplying 8.5 x 7.2 in an attempt to find the area of the trapezoid. Once the decision has been made regarding how to best deconstruct the figure, the students need to apply the Pythagorean Theorem in order to calculate the length of the path contained with the trapezoid.

When this has been calculated, the remaining length and area calculations can be undertaken. The final stage of this multi-step problem is to calculate the cost of the paving based on the basic fee of $200 plus $12 per square foot. This task demands students work across a range of mathematical practices. In particular, they need to make sense of problem and persevere in solving them (MP1) in analyzing the information given and choosing a solution pathway.

Furthermore, students need to attend to precision (MP6) in their careful use of units in the cost calculations. In providing a written rationale of their work, both English learners and native speakers may experience linguistic difficulties in formulating their positions. Additional assistance from the teacher may be required.

The problem below entitled *Baseball Jerseys* addresses the content standards 7.EE.4, 7.NS.3, 8.EE.8, 8.F.4 and Mathematical Practice Standards MP1, MP4, MP7.

Baseball Jerseys. Bill is going to order new jerseys for his baseball team. The jerseys will have the team logo printed on the front. Bill asks two local companies to give him a price. The first company, Print It, will charge $21.50 each for the jerseys. The second company, Top Print, has a set-up cost of $70 and then charges $18 for each jersey. Figure out how many jerseys Bill would need to order for the price from Top Print to be less than from Print It. Explain your answer.

Students may utilize the following approaches in solving this problem: (a) using n for the number of jerseys ordered and c for the total cost in dollars, write an equation to show the total cost of jerseys from Print It; (b) using n to stand for the number of jerseys ordered and c for the total cost in dollars, write an equation to show the total cost of jerseys from Top Print; and (c) use the two equations from the previous two questions to figure out how many jerseys Bill would need to order for the price from Top Print to be less than from Print It.

This problem considers the costing models of two print companies and students should be able to produce two equations $c = 21.5n$ and $c = 70 + 18n$. The third part of this task may be a bit more challenging. Students may construct inequality $70 + 18n < 21.5n$ and then solve for n.

This problem also demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (MP1) by choosing what strategies to use. Students also look for and make use of structure (MP7) in that understanding the properties of linear growth leads to a solution of the problem. Finally, students practice modeling (MP4) because they are being instructed to construct equations.

There are a number of resources available on the Internet that provide grade-level curricular examples aligned to the CCSSM and the Standards for Mathematical Practice. These include department of education sites for other Common Core states. References to these resources can be found throughout this framework. The Math Assessment Resource Services (MARS) Web site provides a multitude of mathematics exercises that specifically focus on the Standards for Mathematical Practice (http://map.mathshell.org/materials/stds.php).

Real World Problems

Teachers do not use real-world situations to serve mathematics; they use mathematics to the serve and address the real-world situations. These problems provide opportunities for mathematics to be learned and engaged in context. Miller (2011) cautions that when we task students with performing real-world math, we do not simply want students to mimic real-world connections; we also want the students to be able to successfully solve associated mathematics problems. Students are already conditioned to do tasks. Even when the task might have strong connections to the real world, it can still just be that: a task to complete. We need to keep this in mind when we ask students to perform real-world math, just as the CCSSM suggest (Miller 2011).

Application of mathematical practices in real world settings and using mathematics to solve real world problems provide ample opportunities for students to develop the 4Cs as described in the Partnership for 21st Century Skills initiative - creativity and innovation, critical thinking and problem solving, communication, and collaboration. Integrating these skills with instruction designed to help students understand and apply mathematical practices is critical for preparing students for college, career, and civic life in the 21st century. Resources connecting the Partnership for 21st Century Skills with the Common Core State Standards can be found at www.p21.org.

In *Exploring World Maps* (California Mathematics Project 2012), adapted from the California Mathematics Project, students work towards mastery of standard 6.PR.3 which calls for the use of ratio and rate reasoning to solve real-world and mathematical examples. The students are provided with the world map and are given Mexico’s surface area (750,000 sq. mi). The students are asked to use this information and other available tools (tracing paper, centimeter grids) to estimate areas of several countries and continents. Finally, the students are asked to provide short-response answers to the following questions: (a) which area did you estimate to be larger, Mexico or Alaska; (b) how many times can Greenland approximately fit into Africa; (c) do you feel confident in your estimations; (d) what estimation methods did you use; (e) now that you know the actual areas (the students are provided with the actual areas prior to answering this question), what surprised you the most; (f) how does the location of equator affect how we see this map.

Once again, the teachers should be cognizant of potential linguistic difficulties that could be experienced by English learners and native speakers alike. Schleppegrell (2007) reminds us that counting, measuring, and other "everyday" ways of doing mathematics draw on "everyday" language, but that the kind of mathematics that students need to develop through schooling uses language in new ways to serve new functions. It is our job to assist all students in acquiring this new language.

Instructional Models

Although the classroom teacher is ultimately responsible for delivering instruction, research on how students learn in classroom settings can provide useful information to both teachers and developers of instructional resources. This section provides an overview of student learning in classroom settings and a number of instructional models for use in the mathematics classroom.

Based upon the diversity of students that is found in California classrooms and the new demands of the CCSSM and the Standards for Mathematical Practice (MP), a combination of instructional models and strategies will need to be considered to optimize student learning. Cooper (2006) lists four overarching principles of instructional design for students to achieve learning with understanding:

1. "Instruction is organized around the solution of meaningful problems.

2. Instruction provides scaffolds for achieving meaningful learning.
3. Instruction provides opportunities for ongoing assessment, practice with feedback, revision, and reflection.
4. The social arrangements of instruction promote collaboration, distributed expertise, and independent learning." (p. 190)

Mercer and Mercer (2005) suggest that instructional models can be placed along a continuum of choices that range from explicit to implicit instruction:

Explicit Instruction	Interactive Instruction	Implicit Instruction
Teacher serves as the provider of knowledge	Instruction includes both explicit and implicit methods	Teacher facilitates student learning by creating situations where students discover new knowledge and construct own meanings.
Much direct teacher assistance	Balance between direct and non-direct teacher assistance	Non-direct teacher assistance
Teacher regulation of learning	Shared regulation of learning	Student regulation of learning
Directed discovery	Guided discovery	Self-discovery
Direct instruction	Strategic instruction	Self-regulated instruction
Task analysis	Balance between part-to-whole and whole-to-part	Unit approach
Behavioral	Cognitive/metacognitive	Holistic

They further suggest that the type of instructional models that will be utilized during a lesson will depend upon the learning needs students in addition to the mathematical content that is being presented. For example, explicit instruction models support practice to mastery, the modeling of skills, and the development of skill and procedural knowledge. On the other hand, implicit models link information to students' background knowledge, develop conceptual understanding and problem solving abilities.

5E Model

Carr and his team (2009) link the 5E Model to three stages of mathematics instruction (introduce, investigate, and summarize). As its name implies, this model is based on recursive cycle of five cognitive stages in inquiry-based learning: (a) engage, (b) explore, (c) explain, (d) elaborate, and (e) evaluate. The role of the teacher in this model is multifaceted. As a facilitator, the teacher nurtures creative thinking, problem solving, interaction, communication, and discovery. As a model, the teacher initiates thinking processes, inspires positive attitudes toward learning, motivates, and demonstrates skill-building techniques. Finally, as a guide, the teacher helps to bridge language gaps and foster individuality, collaboration, and personal growth. The teacher flows in and out of these various roles within each lesson, both as planned and as opportunities arise.

The Three-Phase Model

This model represents a highly structured and sequential strategy utilized in direct instruction. It has proven to be effective for teaching information and basic skills during whole class instruction. In the first phase the teacher introduces, demonstrates, or explains the new concept or strategy, asks questions, and checks for understanding. The second phase is an intermediate step designed to result in the independent application of the new concept or described strategy. In the relatively brief third phase students work independently and receive opportunities for closure. This phase also often serves in part as an assessment of the extent to which students understand what they are learning and how they use their knowledge or skills in the larger scheme of mathematics.

Singapore Math

Singapore math emphasizes the development of strong number sense, excellent mental-math skills, and a deep understanding of place value. It is based on Bruner's principles, a progression from concrete experience—using manipulatives—to a pictorial stage and finally to the abstract level or algorithm. This sequence gives students a solid understanding of basic mathematical concepts and relationships before they start working at the abstract level. Concepts are taught to mastery, then later revisited but not re-taught. The Singapore approach focuses on developing students who are problem solvers. There is a strong emphasis on model drawing, a visual approach to solving word problems that helps students organize information and solve problems in a step-by-step manner. Please visit http://nces.ed.gov/timss/ and http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=WWCIRMSSM09 for additional information.

Concept Attainment Model

Concept attainment is an inductive model to teaching and learning that asks students to categorize ideas or objects by critical attributes. During the lesson teachers provide examples and nonexamples, and then ask students to 1) develop and test hypotheses about the exemplars, and 2) analyze the thinking processes that were utilized. To illustrate, students may be asked categorize polygons and non-polygons that is based upon a pre-specified definition. Through concept attainment, the teacher is control of the lesson by selecting, defining, and analyzing the concept beforehand, and then encouraging student participation through discussion and interaction. This strategy can be used to introduce or strengthen concepts, to review concepts, and in formative assessment (Charles and Senter 2012).

The Cooperative Learning Model

Students working together to solve problems is an important component of the mathematical practice standards. In interactive teaching, students are actively engaged in providing input and assessing their efforts in learning the content. They construct viable arguments, communicate their reasoning, and critique the reasoning of others (MP3). The role of the teacher is to guide students toward the desired learning outcomes. The cooperative learning model involves students working either in partners or in mixed-ability groups to complete specific tasks. It assists teachers in addressing the needs of the wide diversity of students that is found in many classrooms. The teacher presents the group with a problem or a task and sets up the student activities. While the students work together to complete the task, the teacher monitors progress and assists student groups when necessary (Charles and Senter 2012; Burden and Byrd 2010).

Cognitively Guided Instruction (CGI)

This model of instructions calls for the teacher asking students to think about different ways to solve a problem. A variety of student-generated strategies are used to solve a particular problem such as: using plastic cubes to model the problem, counting on fingers and using knowledge of number facts to figure out the answer. The teacher then asks the students to explain their reasoning process. They share their explanations with the class. The teacher may also ask the students to compare different strategies. Students are expected to explain and justify their strategies, and along with the teacher, take responsibility for deciding whether a strategy that is presented is viable.

This instructional model puts more responsibility on the students. Rather than simply being asked to apply a formula to several virtually identical math problems, they are challenged to find their own solutions. In addition, students are expected to publicly explain and justify their reasoning to their classmates and the teacher. Finally, teachers are required to open up their instruction to students' original ideas, and to guide each student according to his or her own developmental level and way of reasoning.

Expecting students to solve problems with strategies that haven't been taught to them and asking students to explain and justify their thinking has a major impact on students' learning. Students who develop their own strategies to solve addition problems are likely to intuitively use the commutative and associative properties of addition in their strategies. Students using their own strategies to solve problems and justifying these strategies also contributes to a positive disposition toward learning mathematics

(Wisconsin Center for Education Research 2007; National Center for Improving Student Learning and Achievement in Mathematics and Science 2000.)

Problem-Based Learning

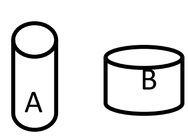
The Standards for Mathematical Practice emphasize the importance of: making sense of problems and persevering in solving them (MP 1); reasoning abstractly and quantitatively (MP 2); and solving problems that are based upon "everyday life, society, and the workplace" (MP4). Implicit instruction models such as problem-based learning, project-based learning, and inquiry-based learning provide students with the time and support to successfully engage in mathematical inquiry by collecting data and testing hypotheses. Burden and Byrid (2010) attribute John Dewey's model of reflective thinking for the basis of this instructional model: "(a) identify and clarify a problem; (b) form hypotheses; (c) collect data' (d) analyze and interpret the data to test the hypotheses; and (e) draw conclusions" (p. 145). These researchers suggest two different approaches can be utilized to problem-based learning. During *guided inquiry*, the teacher provides the data and then questions the students in an effort for them to arrive at a solution. Through *unguided inquiry*, students take responsibility for analyzing the data and coming to conclusions.

In problem-based learning, students work either individually or in cooperative groups to solve challenging problems with real world applications. The teacher poses the problem or question, assists when necessary, and monitors progress. Through problem-based activities, "students learn to think for themselves and show resourcefulness and creativity" (Charles and Senter 2012, 125). Nevertheless, Martinez (2010, 149) cautions that when students engage in problem solving they must be allowed to make mistakes: "If teachers want to promote problem solving, they need to create a classroom atmosphere that recognizes errors and uncertainties as inevitable accoutrements of problem solving." Through class discussion and feedback, student errors become the basis of furthering understanding and learning (Ashlock 1998). Please see Modeling in the appendix for additional information.

Scientific Inquiry Model

Scientific inquiry is a methodology of science that includes observing, measuring, inferring, predicting, classifying, hypothesizing, experimenting and concluding. Inquiry also refers to the instructional approaches that enable teachers to teach concepts through exploration. It is built around intellectual confrontations. The student is presented with a puzzling situation and uses inquiry skills to solve it. The ultimate goal is to have the students experience the creation of new knowledge. Many students regard learning mathematics as a matter of memorizing formulas. The inquiry approach demonstrates the logic process of formulating rules. Please consider example below:

a. *Phase One: Confrontation*. Two different sizes of cylindrical cups are available at an All-You-Can Eat restaurant, one with a wider base and the other with a greater height. Which cup contains more water? Show students the cups.



b. *Phase Two: Prediction*. Students engage in small group discussion to predict which cup will hold the most water. During whole group discussion, the teacher respects and records all of the student's predictions and reasoning. They may choose cup A, cup B, or decide that cup A and B can hold the same amount of water.

c. *Phase Three: Experimentation*. Ask students for a strategy to verify their predictions. The most common suggestion is to fill both cups with water.

d. *Phase Four: Analysis and Generalization*. Isolate relevant variables and hypothesize (and test) causal relationships. While the inquiry task is to find the cup with greater volume, the goal of this activity is to generalize that volume is the product of base area and height.

This is just a sampling of the multitude of instructional models that have been researched across the globe. Ultimately, teachers and administrators must determine what works best for their student populations. Teachers may find that a combination of several instructional approaches is appropriate in any given classroom.

Instructional Strategies for the Mathematics Classroom

As teacher progress through their career they develop a repertoire of instructional strategies to convey new information to their students. The following section discusses a several instructional strategies but certainly is not an exhaustive list. Teachers are encouraged to seek out other math teachers, professional learning from county offices of education and other providers, as well as research the Web to build their repertoire.

Using Discourse in the Mathematics Classroom

The CCSSM and Standards for Mathematical Practice expect students to demonstrate competence in making sense of problems (MP 1); constructing viable arguments (MP 3); and modeling with mathematics (MP 4). In other words, students will be expected to communicate their understanding of mathematical concepts, receive feedback, and progress to deeper understanding. Ashlock (1998, 66) concludes that when students communicate their mathematical learning through discussions and writing, they are able to "relate the everyday language of their world to math language and to math symbols." Van de Walle (2007, 86) adds the process of writing enhances the thinking process by requiring students to collect and organize their ideas. Furthermore, as an assessment tool, student writing "provides a unique window to students' thoughts and the way a student is thinking about an idea."

Number / Math Talks. Parrish (2010) describes number talks as "classroom conversations around purposefully crafted computation problems that are solved mentally. The problems in a number talk are designed to elicit specific strategies that focus on number relationships and number theory. Students are given problems in either a whole-or small-group setting and are expected to mentally solve them accurately, efficiently, and flexibly. By sharing and defending their solutions and strategies, students have the opportunity to collectively reason about numbers while building connections to key conceptual ideas in mathematics. A typical classroom number talk can be conducted in five to fifteen minutes."

During a number talk, the teacher writes a problem on the board and gives students time to solve the problem mentally. Once students have found an answer, they are encouraged to continue finding efficient strategies while others are thinking. They indicate that they have found other approaches by raising another finger for each solution. This quiet form of acknowledgement allows time for students to think, while the process continues to challenge those who already have an answer. When most of the students have indicated they have a solution and strategy, the teacher calls for answers. All answers – correct and incorrect – are recorded on the board for students to consider.

Next, the teacher asks a student to defend their answer. The student explains their strategy and the teacher records the students thinking on the board exactly as the student explains it. The teacher serves as the facilitator, questioner, listener, and learner. The teacher then has another student share a different strategy and records their thinking on the board. The teacher is not the ultimate authority, but allows the students to have a "sense of shared authority in determining whether an answer is accurate".

Questions teachers can ask:

- How did you solve this problem?
- How did you get your answer?

- How is Joe’s strategy similar or different than Leslie’s strategy?

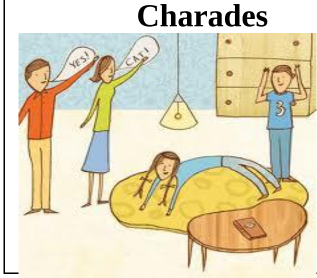
5 Practices for Orchestrating Productive Mathematics Discussions. Smith and Stein (2011) identify 5 practices that assist teachers in facilitating instruction that advances the mathematical understanding of the class:

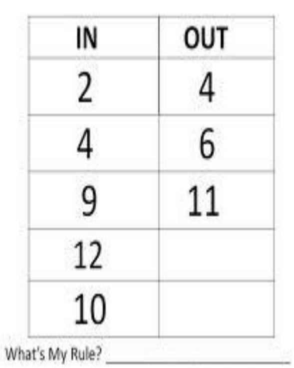
- Anticipating
- Monitoring
- Selecting
- Sequencing
- Connecting

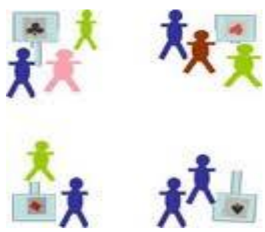
Organizing and facilitating productive mathematics discussions for the classroom take a great deal of preparation and planning. Prior to giving a task to the students, the teacher should anticipate the likely responses that students will have so that they are prepared to serve as the facilitator of the lesson. Students will usually come up with a variety of strategies, but it is helpful when leading the discussion if you have already anticipated some of them. The teacher then poses the problem and gives the task to the students. The teacher monitors the student responses while they work individually, in pairs, or in small groups. The teacher pays attention to the different strategies that students are using. In order to conduct the share and summarize portion of the lesson, the teacher selects student to present their mathematical work and sequences the sharing so that the various strategies are presented in a specific order in to highlight the mathematics of the instructional goal. As the teacher conducts the discussion, the teacher is intentional about asking questions to facilitate students connecting the responses to the key mathematical ideas.

Student Engagement Strategies

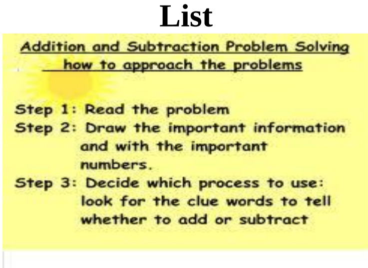
Building a robust list of student engagement strategies is essential for all teachers. When students are engaged in the classroom, they remain focused and on-task. This also provides for good classroom management and effective teaching and learning. The table below provided by the Rialto Unified School District illustrates several student engagement strategies for the mathematics classroom:

Student Engagement Strategies	**Description**	**Math Example**
Appointment Clock 	Partnering to make future discussion/work appointments. (good grouping strategy)	Student are given a page with a clock printed on it that they use to set appointment times to meet with other students to discuss math problems.
Carousel-Museum Walk 	Each group posts sample work on the wall and the leader for that group stands near the work, as the rest of the group rotates around the room, looking at all the samples.	Each group is given a poster paper & Math problem to work on. Once the groups are finished, paper is posted on the walls around the classroom. The leader stays with the poster to explain the work, while the other students walk around the room looking at the other students' work.
Charades 	Students individually, or with a team, act out a scenario.	Students work in teams to act out word problems while others try to solve the problem.

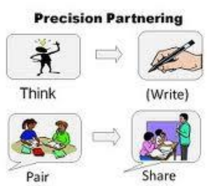
Clues (Barrier Games) 	One partner has a picture of information the other student does not have. Sitting back-to-back or using a visual barrier, students communicate to complete the task.	Working in teams of 2, each student has a different problem to communicate to the other student, who is to try and solve the problem from the information provided by the first student. The students sit with a barrier between them during the activity.
Coming to Consensus 	Sharing their individual ideas, the group comes to a consensus to share with the whole class.	Each member of the group shares their answer to a given problem, the steps they used etc. When the group comes to a consensus, they share out with the whole class.
Explorers & Settlers 	Assign half the students to be explorers and half settlers. Explorers seek out a settler to discuss a question. Students can change roles and repeat process.	Half of the students are explorers who have a Math term or problem. The other half is settlers who have the definitions or answers. Explores seek out the settler with the correct answers and discuss the information.
Find My Rule  IN | OUT 2 | 4 4 | 6 9 | 11 12 | 10 | What's My Rule? ____________	Using cards, students are given cards and must find the person that matches their card. One person has a card with a rule, and the other has an example of that rule, as they find their partner.	A great strategy for inductive/deductive reasoning. Works well for grouping students randomly and developing problem-solving skills. Cards are prepared one with a problem and the other with the "rule." Students circulate throughout the room to match the cards that are connected or related by the "rule." Once all members of the group have been found, group

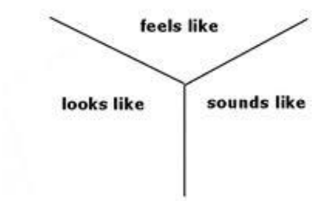
		members will articulate the rule and how the group is connected.
Find Your Partner 	Each student is given a card that matches another student's card in some way.	Examples: Math problem with steps to solution Concept + example
Four Corners 	Assign each corner of the room a category related to a topic. Students write which category they are most interested in, giving reasons, and then form groups in those corners.	Students are divided in 4 groups and sent to a corner which is numbered 2 - 5 Teacher then asked a problem with the answer being a multiple of 2 – 5. Students in a corner that is a factor of that number will move to another corner. If teacher calls out 6, students in corners labeled 2 and 3 will move the activity ends with a prime number answer and students return to their seats.
Give One, Get On 	After brainstorming ideas, students circulate among other students sharing one idea and getting one. Students fold paper lengthwise they label the left side "give one" and the right side "get one"	Teacher gives the class a multi-step problem to solve and a time limit. On the right side they list all the steps they know before finding a partner. Partner A gives an answer to B. If Partner B has that answer, they check it off. If it's a new answer, they write it on the "GET ONE" side & repeat the process for Partner B.

		Once both partners have exchanged ideas, they put their hands up, find new partners, and continue until teacher says to stop.
Inside Outside Circle 	Two concentric circles of students stand or sit, facing one another. The teacher poses a question to the class, and the partner responds. At a signal, the outer of inner circle or outer circle rotates and the conversation continues.	Students share information & problem solve. Teacher prepare question cards for each student One student from each pair moves to form one large circle facing outward the other students find and face their partners forming two concentric circles. Inside circle students ask a question from their card, outside students answer then they discuss the problem before switching roles. Once both students have asked & answered a question, the inside circle rotates clockwise to a new partner.
Jigsaw 	Group of students assigned a portion of a text, teach that portion to the remainder of the class.	"Factoring Jigsaw," in which each student becomes an expert on a different concept or procedure in the factoring process and then teaches that

		concept to other students.
KWL  	Cognitive graphic organizer and sets the stage for learning.	Math teachers use as a diagnostic tool to determine student readiness, using pre-test questions and a KWL chart the teacher asks students to identify what they already **K**now, what they **W**ant to know, and what they need to do to **L**earn.
Line Up (class building) 	Students line up in a particular order given by teacher e.g. alphabetically by first name, by birth date, shortest to tallest, etc. Students talk to a partner sharing how they feel about their position in the line-up.	Students line up in order by the square root or multiples of a given number. Once in line, they share how they feel about their position in the line-up, and explain how found their place. (good activity for the first day of class).
Making A List  	Two students, using one word or phrase add items to a list.	Student could have a multi-step or word problem and list the steps needed to solve the problem
Numbered Heads Together 	Each student, within a group is assigned a number. Teacher gives a question or assignment and after students are given time to independently answer the question.	Good strategy for grouping students to work in specific ability level groups. Teacher assigns student numbers then assigns each number group a problem at their level. Students then work together or independently to answer the problem.

Partner Up 	A strategy used to find a partner to engage with.	Good activity for students to find a partner to study with for an upcoming test.
Quiz-Quiz Trade  	Using two-sided, pre-made cards, students in pairs quiz each other, trade cards and then find another partner.	Can be used to help students review Math vocabulary, math facts or improve their mental math skills.
Socratic Seminar 	A group of students participate in a rigorous, thoughtful dialogue, seeking deeper understanding of complex ideas. Guidelines and language strategies are taught and followed during the seminar.	A Socratic seminar with a wingman formation works well for Math. Start with student s sitting in 2 concentric circles. Two outer circle students sit behind one inner circle as their “wingmen”, becoming a team. The inner circle participates in the discussion, and the outer circle students listen and takes notes. Frequently the teacher stops the discussion for the teams to share their ideas then continues.
Talking Sticks 	In teams, each member takes a turn and places their stick in the center of the team to talk about a given topic.	Good for working in teams on projects to ensure that all group members have a turn to participate in the group’s discussion.

Team Share Out 	Teams take turns sharing out their final product.	Students are working in teams on different problems. After solving the problem, each team has the opportunity to share their answer with the whole class.
Think Pair Share 	Partners face each other, given the amount of time and topic, take turns talking.	Could be used for students to discuss how they found their answer to the daily bell-work to help change things up and encourage student engagement.
Think-Write-Pair-Share 	Given a short amount of time, students write their ideas about a given topic and share their ideas in pairs.	Students are given a word problem to solve. First, they have a set amount of time to think about how to solve it. Then, they write the steps it would take to solve the problem. Finally, students share their ideas with a partner.
Whip Around 	In a group, each person shares their ideas with the whole group, from a given topic.	Could work with solving word problems. Each student would share their ideas on how they would solve the problem - What steps would you use?

Wrap Around 	After students write their ideas about a topic, each student shares one idea, repeating the statement of the previous student.	Teacher gives the whole class a problem then allows the students time to write the steps on how to solve the problem before having each student share out the one step in the process.
Y-Chart 	A graphic organizer created by a group of students to recognize what something "feels like, sounds like, and looks like"	Use as a graphic organizer to help students organize their thoughts and ideas. Can also be used to set up lab expectations.

Tools for Mathematics Instruction

There are a number of mathematical instructional tools that teachers can use to make mathematics concepts more concrete for their students. This is especially important in classrooms with a large number of English Learners or students with disabilities. This section highlights a small number of the tools that teachers can use with their students.

Visual Representations. The Mathematical Practice Standards suggest that students look for and make use of structure (MP7); construct viable arguments (MP 3); model with mathematics (MP4) and use appropriate tools strategically (MP 5). Visual representations can be utilized in obtaining proficiency with these standards when used in alignment with the content standards. In order to develop understanding, mathematical concepts should not be taught in isolation. Instead, meaningful relationships that connect superordinate and subordinate concepts should be identified.

Diagrams, concept maps, graphic organizers, and flow charts can be utilized to show relationships (Martinez 2010). Burden and Byrd (2010) write that visual representations such as graphic organizers combine the use of words and phrases with symbols such as arrows to represent relationships. Ashlock (1998) posits that concept maps can be utilized as an overview to the lesson, to summarize what has been taught, and to inform instruction and recommends that these representations are well suited to chart out computational procedures, and can be created by teachers as well as by students. Visual representations may also be through drawings (e.g., students draw simple pictures to illustrate a story problem); and charts (e.g., fractions and decimals can be sorted and grouped into categories such as greater than one half, one half, and less than one half).

Advanced Organizers. In order for understanding to occur, new knowledge should be connected “meaningfully” to knowledge that students already possess: “When connections are minimal and superficial, we sometimes call the product rote learning. But when connections are rich and the new knowledge integrates with prior knowledge to produce a coherent picture, the result is understanding” (Martinez 2010, 74). Miller (2011) advocates the use of advanced organizers to provide background information, review previous content, provide a lesson overview, and motivate students.

Concrete Models. The Mathematical Practice Standards advocate the use of concrete models in order that students make sense of problems and persevere in solving them (MP1); and use appropriate tools strategically (MP5). Martinez (2010, 229) suggests that learning that utilizes different modes of instruction is necessary to promote both student understanding and recall from long-term memory: “Good teachers know that presenting ideas in a variety of ways can make instruction more effective and more interesting, as well as better able to reach a variety of learners.” Concrete models such as manipulatives can be utilized to help students learn a wide range of mathematical concepts. For example, students create models to demonstrate the Pythagorean Theorem, they utilize tiles to demonstrate an algebra expression, and they use base ten models to demonstrate complex computational procedures.

There are a multitude of instructional resources available for teachers of mathematics. It would not be possible to capture them all in this chapter. For example, San Diego Unified School District offers an exhaustive list of mathematics instructional “routines” at http://www.sandi.net/Page/33501. Teachers are encouraged to seek out multiple sources of information and research to build their instructional repertoire.

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